Last time: - π-systems, d-systems

- d-maps

Today: integration using a measure,

L^p space

Lemma: Let X be a set. A collection $\mathcal{M} \subseteq \mathcal{P}(X)$ is a σ-algebra iff it satisfies the following

1) $X \in \mathcal{M}$
2) If $E \in \mathcal{M}$ then $E^c \doteq X \setminus E \in \mathcal{M}$
3) If $E_1, E_2, \ldots$ is any sequence of sets in $\mathcal{M}$ then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$

Furthermore, if $\mathcal{M}$ is a σ-algebra any finite or countable union or intersection of sets in $\mathcal{M}$ will also be in $\mathcal{M}$.

Def: A measurable space consists of a set X and a σ-algebra $\mathcal{M}$ on X.

Elements of $\mathcal{M}$ are called measurable sets.

Def: A (positive) measure on a measurable space $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that

1) $\mu(\emptyset) = 0$

σ-additivity 2) $\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$

for any pairwise disjoint sequence $E_1, E_2, \ldots$

E.g. Let X be any set

and $\mathcal{M} = \mathcal{P}(X)$. Then the counting measure on X is

$$\mu : \mathcal{P}(X) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$$
$$E \longmapsto |E|$$

cardinality of E

Def: If $(X, \mathcal{M}_1)$ and $(Y, \mathcal{M}_2)$ are two measurable spaces, then a measurable function $f : X \to Y$ is a function such that if $E \in \mathcal{M}_2$ then $f^{-1}E \in \mathcal{M}_1$. Equivalently, f^{-1} is a σ-map $\mathcal{M}_2 \to \mathcal{M}_1$.

We want to define a measure on $\mathbb{R}$. Essentially, want to assign a size to subsets of $\mathbb{R}$.

We want to build on the intuition that the size of $[a,b]$ is $b-a$.

Define the length of an interval to be

$$\ell([a,b]) = \ell((a,b)) = \ell([a,b)) = \ell((a,b]) = b-a.$$

Def: A measure on a group G is said to be translation invariant if for any measurable set E and any $g \in G$, the set

$$gE = \{gh \mid h \in E\}$$

is measurable and

$$\mu(gE) = \mu(E).$$

Why do we introduce σ-algebras

when we are doing measure theory?
Why not define a measure on all of $\mathcal{P}(\mathbb{R})$?

$\rightsquigarrow$ No way to do this consistently (at least assuming translation invariance)

Counterexamples: Vitali set,
Banach-Tarski paradox

So we can't define a measure on every set. But we'll at least want to measure open and closed sets, their (countable) unions and intersections, and complements.

Def: Let X be a topological space with τ = the collection of open sets.

The minimal σ-algebra containing τ is called the σ-algebra of Borel sets $\mathcal{B}_X$.

So a Borel set is one that can be constructed from open sets via a sequence of operations of the form

- taking complement
- taking countable union.

Def: The Lebesgue (or Haar or Borel) measure on $\mathbb{R}^n$ is defined on Borel sets in $\mathbb{R}^n$ by

$$\mu(E) = \inf \left\{ \sum_{i=1}^{\infty} \ell(R_i) \,\middle|\, R_1, R_2, \ldots \text{ are rectangles in } \mathbb{R}^n \right.$$

$$\text{st } E \subseteq \bigcup_i R_i \Big\}$$

where a rectangle is a set of the form
$$R = (a_1, b_1) \times (a_2, b_2) \times \dots \times (a_n, b_n),$$
and $\ell(R) = (b_1 - a_1) \cdot (b_2 - a_2) \cdot \dots \cdot (b_n - a_n)$.

In particular, for $\mathbb{R}$,
$$\mu(E) = \inf \Big\{ \sum_{i=1}^{\infty} \ell(I_i) \;\Big|\; I_1, I_2, \dots \text{ intervals st } E \subseteq \bigcup_i I_i \Big\}.$$

We can prove this is a measure using Caratheodory's criterion.

The Lebesgue measure is the unique measure on Borel sets which is translation invariant and satisfies

1) $\mu([0,1]) = 1$

2) $\mu(E) = \inf\left\{\mu(U) \mid E \subseteq U, \; U \text{ open}\right\}$

Once we have a measure space $(X, \mathcal{M}, \mu)$, we can define integration!

Def: A simple function on X is a function $f: X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ of the form $f = \sum_{i=1}^{r} a_i \mathbb{1}_{E_i}$,

where $a_1, \ldots, a_r \geq 0$ and where $\mathbb{1}_{E_i}$ is the characteristic function of E_i, defined by

$$x \mapsto \begin{cases} 1 & \text{if } x \in E_i \\ 0 & \text{if } x \notin E_i. \end{cases}$$

If f is a simple function, then

$$\int_X f\,d\mu \doteq \sum_{i=1}^{r} a_i\,\mu(E_i) .$$

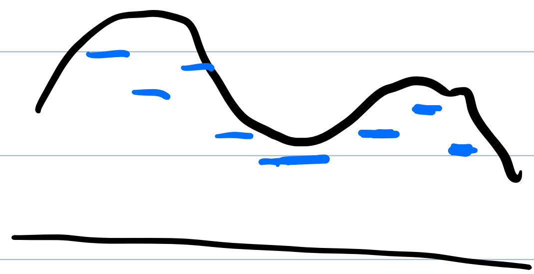
need to check this is independent of representation as simple function

Now let $f: X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be any measurable function (with the Borel σ-algebra on $\mathbb{R}_{\geq 0} \cup \{\infty\}$).

The Lebesgue integral of f is

$$\int_X f\,d\mu = \int_X f(x)\,d\mu(x)$$

$$\doteq \sup\left\{ \int_X \phi\,d\mu \;\middle|\; \phi \leq f \text{ pointwise and } \phi \text{ is a simple function} \right\}$$



If $f: X \to \mathbb{R}$ is measurable,

we take $f_+ : x \mapsto \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$

and $f_- : x \mapsto \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geq 0 \end{cases}$

so that $f = f_+ - f_-$.

and define $\int_X f \, d\mu := \int_X f_+ \, d\mu - \int_X f_- \, d\mu$.

Similarly can define for $f: X \to \mathbb{C}$.

We say f is integrable if

$\int_X |f| \, d\mu < \infty$

Sunday we'll define L^p spaces and discuss dominated convergence.